Quantitative Analysis

The Royal Bank of Scotland

Extending mean-variance asset allocation to incorporate put and call options

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Contacts:
Paul Doust
Global Head of Quantitative Analysis
+44 20 70856015
paul.doust@rbs.com

www.rbsmarkets.com

- A mean-variance result is presented for portfolios containing vanilla options. The main result is an analytic formula for the optimal payoff function $f(S)$ which maximises the Sharpe ratio for a position in a single stock S. The main inputs to the formula are the risk-neutral probability distribution $\rho_{RN}(S)$ which is used in option pricing, and the real-world probability distribution $\rho_{RW}(S)$ associated with Markowitz mean-variance. The result is completely general and applies to any probabilty distributions for S, not just normal or log-normal distributions.
- An equivalent Markowitz style formula which involves options directly is also derived. This shows how to convert the optimal payoff function $f(S)$ into a practical option position.
- Taking the UK FTSE index as an example with data from July 2007, it is shown that the payoff which maximises the Sharpe ratio can be well approximated by a position in the index itself together with a long position in a 15% delta put option and a short position in a 15% delta call option. This provides downside protection on the index, as well as giving a higher Sharpe ratio.
- Going beyond mean and variance, it is shown that analytic formulas can also be found when an aversion to the skewness or the kurtosis of $\rho_{RW}(S)$ is incorporated into the framework. However a consideration of the probability distribution of the optimal payoff suggests that optimising moments of $\rho_{RW}(S)$ may not be appropriate when considering options. Nonetheless, the analytic formula produced by mean-variance does behave well in the example cases considered.

1 Introduction

Traditional mean-variance[1] applies to portfolios of assets where the returns are driven by separate stochastic processes which are usually correlated. Put and call options on an asset, however, are derivative assets whose values are driven by the same stochastic process that drives the underlying asset. This article extends basic Markowitz mean-variance to take account of these kinds of assets in the portfolio selection process.

The layout of this article is as follows: Section 2 is a short recap of basic Markowitz mean-variance. Section 3 discusses the properties of the two probability distributions $\rho_{RN}(S)$ and $\rho_{RW}(S)$ which are required for this work. The main result is then derived in section 4, followed by a Markowitz-style discrete approximation to the result in Section 5. Although the risk measure used here is variance, it will be seen that the result is sensitive to the entire probability distribution, not just the second central moment. The results are illustrated in section 6, first for log-normal distributions, and then for another couple of distributions including an example taken from the UK FTSE index. Section 7 discusses issues beyond the mean-variance result of section 4. To show that an analytic formula is possible when considering an aversion to some higher moments of $\rho_{RW}(S)$, section 7.1 re-derives the main result when the investor has an aversion to the skewness of $\rho_{RW}(S)$ as well as the variance. Section 7.2, however, discusses issues raised by a consideration of the probability distribution of the optimal payoff $f(S)$, which raises doubts about the main result of section 4. Nonetheless, the main result seems to have all the right properties, and this is discussed in the conclusion in section 8. The appendix starting on page 18 gives explicit formulas for the results of sections 4 and 5 for the cases of normal and log-normal distributions.

2 Basic Markowitz mean-variance

The mean-variance portfolio optimisation technique developed by Markowitz[1] calculates the optimal weights for assets in a portfolio by assuming that the asset returns R_i are jointly normally distributed. To be explicit, parameterise the distribution of R_i by the expected excess returns μ_i and the covariance matrix Σ_{ij} so that

$$\mu_i = E(R_i - r_f) \ , \quad \Sigma_{ij} = \sigma_i \sigma_j \rho_{ij} = E\left((R_i - E(R_i))(R_j - E(R_j))\right) \ , \quad (1)$$

where r_f is the risk-free simple interest rate. Using matrix notation, let $\boldsymbol{\mu}$ be the column vector of μ_i, and let $\boldsymbol{\Sigma}$ be the square matrix Σ_{ij}. To derive mean-variance, look for $\mathbf{w}$ to maximise the quadratic utility function

$$U(\mathbf{w}) = \boldsymbol{\mu}'\mathbf{w} - \frac{1}{2\lambda}\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} \quad , \quad (2)$$

where λ is the scalar risk-tolerance parameter. In the absence of any constraints, $U(\mathbf{w})$ is maximised by choosing

$$\mathbf{w} = \lambda \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} \quad , \quad (3)$$

This article will replicate this analysis for the case where the investable assets are a single stock together with all European put and call options on that stock which expire at the investment horizon.

3 Modelling the outcomes for a single stock

The mean-variance result derived below depends on the relationship between two probability distributions for the possible stock prices S at the investment horizon:

- The risk-neutral probability distribution $\rho_{RN}(S)$, which is the basis for option pricing at the start of the investment period;
- The real-world probability distribution $\rho_{RW}(S)$, which is the basis for mean-variance asset allocation.

3.1 The risk-neutral probability distribution $\rho_{RN}(S)$

Put and call options which expire on the investment horizon, where the underlying is the stock S, are priced using the risk-neutral probability distribution $\rho_{RN}(S)$. One of the key features of $\rho_{RN}(S)$ is that the mean of the distribution is the forward stock price F. Assuming that the stock pays no dividends during the investment period, F can be calculated directly from the stock price S_0 at the start of the investment period and the risk free rate r_f, so that

$$F = S_0 \left(1 + r_f \tau\right) = E_{RN}\left(S\right) = \int S \rho_{RN}\left(S\right) dS \quad , \tag{4}$$

where τ is the time from the start to the end of the investment period. Note that simple interest rates are being used here, to keep compatibility with the results of section 2.

Write the payoff for call options and put options at strike K using the the following notation

$$\text{Call option payoff} \quad = \max\left(S - K, 0\right) = \left(S - K\right)\ \theta\left(S - K\right) = \left(S - K\right)^{+} \tag{5}$$

$$\text{Put option payoff} \quad = \max\left(K - S, 0\right) = \left(K - S\right)\ \theta\left(K - S\right) = \left(K - S\right)^{+} \tag{6}$$

where $\theta\left(x\right)$ is the Heaviside step function defined by

$$\theta\left(x\right) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{cases} \quad . \tag{7}$$

With this notation, put-call parity corresponds to the identity

$$\left(S - K\right)^{+} - \left(K - S\right)^{+} = S - K \quad . \tag{8}$$

Then at the start of the investment period, the call option prices $C\left(K\right)$ and the put option prices $P\left(K\right)$ can be written

$$C\left(K\right) = \frac{E_{RN}\left(\left(S - K\right)^{+}\right)}{\left(1 + r_f \tau\right)} = \frac{\int \left(S - K\right)^{+} \rho_{RN}\left(S\right) dS}{\left(1 + r_f \tau\right)} \quad , \tag{9}$$

$$P\left(K\right) = \frac{E_{RN}\left(\left(K - S\right)^{+}\right)}{\left(1 + r_f \tau\right)} = \frac{\int \left(K - S\right)^{+} \rho_{RN}\left(S\right) dS}{\left(1 + r_f \tau\right)} \quad . \tag{10}$$

The distribution $\rho_{RN}\left(S\right)$ is known if all the option prices $C\left(K\right)$ are observable in the market. Differentiating (9) twice with respect to K shows that

$$\rho_{RN}\left(S\right) = -\left(1 + r_f \tau\right) \frac{\partial^2 C\left(K\right)}{\partial K^2} \quad . \tag{11}$$

3.2 The real-world probability distribution $\rho_{RW}\left(S\right)$

In the real-world, a return μ_S is expected for holding a risky asset S, where μ_S is defined as the return in excess of the risk-free rate r_f. This defines the mean of the real-world probability distribution $\rho_{RW}\left(S\right)$ so that

$$S_0 \left(1 + \left(r_f + \mu_S\right) \tau\right) = F + S_0 \mu_S \tau = E_{RW}\left(S\right) = \int S \rho_{RW}\left(S\right) dS \quad . \tag{12}$$

In general nothing else is known about $\rho_{RW}\left(S\right)$. However a convenient assumption will be that $\rho_{RW}\left(S\right)$ has the same form as $\rho_{RN}\left(S\right)$, except that the mean is shifted from F to $F + S_0 \mu_S \tau$, and in what follows this will be regarded as the benchmark case for an investor to consider. Nonetheless, the main result in equation (20) below does not depend on this assumption.

Investors are concerned with real-world expected returns and real-world risk, so mean-variance is formulated with $\rho_{RW}\left(S\right)$ rather than $\rho_{RN}\left(S\right)$.

4 Mean-variance incorporating put and call options

Consider asset allocation for a time period τ, choosing between a single stock and all possible put and call options on that stock which expire on the investment horizon after time τ.

It is possible to formulate the mean-variance problem in terms of a holding in cash, stock, and put and call options, and indeed this will be done in section 5 below. However to derive a simple analytic result, it turns out to be easier to look for the optimal payoff function $f(S)$ on the investment horizon instead. In this context, the utility function $U(\mathbf{w})$ from (2) will become a functional $U[f]$.

If the optimal payoff $f(S)$ can be determined, then it is straightforward to translate $f(S)$ into an equivalent holding of cash, stock, and put and call options using the following identity:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \int_{x_0}^{\infty} f''(y)(x-y)^+ \, dy + \int_{-\infty}^{x_0} f''(y)(y-x)^+ \, dy \quad . \tag{13}$$

Equation (13) is true for any continuous payoff function $f(x)$. For example, if $f(x) = (x-K)^+$, then $f'(x) = \theta(x-K)$ and $f''(x) = \delta(x-K)$, and it is straightforward to verify that (13) is true for all x_0. To convert $f(S)$ into positions in the stock and options on the stock, it makes sense to choose x_0 to be the forward F. Then $f'(x_0)$ determines the holding of the stock itself, with $f''(x)$ determining the positions in the put and call options. The option holding is unique up to changes allowed by put-call parity (8).

Because the risk-neutral distribution $\rho_{RN}(S)$ is used to price the payoff at the start of the investment period, the initial portfolio value P_0 is given by

$$P_0 = \frac{\int f(S)\,\rho_{RN}(S)\,dS}{(1 + r_f\tau)} \quad . \tag{14}$$

Hence on the investment horizon, the excess value over the return provided by the risk free rate is given by

$$\int \frac{f(S)}{P_0}\rho_{RW}(S)\,dS - (1 + r_f\tau) = \int \frac{f(S)}{P_0}\left(\rho_{RW}(S) - \rho_{RN}(S)\right)dS \quad , \tag{15}$$

so that the functional $U[f]$ to maximise is given by

$$\begin{aligned} U[f] = &\int \frac{f(S)}{P_0}\left(\rho_{RW}(S) - \rho_{RN}(S)\right)dS \\ &- \frac{1}{2\lambda P_0^2}\left(\int f(S)^2 \rho_{RW}(S)\,dS - \left(\int f(S)\,\rho_{RW}(S)\,dS\right)^2\right) \quad . \end{aligned} \tag{16}$$

To maximise $U[f]$, the first order derivative condition to satisfy with respect to $f(S)$ is

$$\rho_{RW}(S) - \rho_{RN}(S) - \frac{1}{\lambda P_0}\left(f(S)\,\rho_{RW}(S) - \rho_{RW}(S)\int f(S)\,\rho_{RW}(S)\,dS\right) = 0 \quad . \tag{17}$$

Rearranging (17) shows that $f(S)$ can be written

$$f(S) = E_{RW}(f(S)) + P_0\lambda\left(1 - \frac{\rho_{RN}(S)}{\rho_{RW}(S)}\right) \quad , \tag{18}$$

$$\text{where} \quad E_{RW}(f(S)) = \int f(S)\,\rho_{RW}(S)\,dS \quad . \tag{19}$$

In (18), $E_{RW}(f(S))$ should be regarded as a constant to be determined so that the initial portfolio value is P_0. Hence the optimal $f(S)$ can be written

$$f(S) = C - P_0 \lambda \frac{\rho_{RN}(S)}{\rho_{RW}(S)} \quad , \tag{20}$$

where the constant C is given by

$$C = P_0 (1 + r_f \tau) + P_0 \lambda \int \frac{\rho_{RN}(S)^2}{\rho_{RW}(S)} dS \quad . \tag{21}$$

Equation (20) is the major result of this article and its implications will be discussed in detail below. Furthermore, section 7.1 extends the result when a skewness aversion term is added to $U[f]$. For now, note that if $\rho_{RN}(S) = \rho_{RW}(S)$, no additional return is expected above the risk-free rate and in this situation the optimal allocation is just cash because $f(S)$ given by (20) is a constant.

With the optimal $f(S)$ from (20), the expected excess return is given by

$$\text{Expected excess return} \quad = \frac{E_{RW}(f(S))}{P_0} - (1 + r_f \tau) = \lambda \left(\int \frac{\rho_{RN}(S)^2}{\rho_{RW}(S)} dS - 1 \right) \quad , \tag{22}$$

and the variance of the return is given by

$$\begin{aligned} \text{Variance of return} \quad &= \int \rho_{RW}(S) \left(\frac{f(S)}{P_0} - \frac{E_{RW}(f(S))}{P_0} \right)^2 dS \\ &= \lambda^2 \left(\int \frac{\rho_{RN}(S)^2}{\rho_{RW}(S)} dS - 1 \right) \quad . \end{aligned} \tag{23}$$

Hence the Sharpe ratio at the optimal $f(S)$ is given by

$$\text{Sharpe ratio} \quad = \sqrt{\int \frac{\rho_{RN}(S)^2}{\rho_{RW}(S)} dS - 1} \quad . \tag{24}$$

As with traditional mean-variance allocation, it is easy to show that (24) is the maximum achievable Sharpe ratio of any payoff function at the investment horizon.

Note that although the variance was the risk measure used to derive (20), the result is sensitive to the entire distribution function and not just the second central moment. Explicit formulas for (20) are given in the appendix on page 18 for the cases when the probability distributions are normal and log-normal.

5 A discrete formulation in terms of put and call options

As mentioned in the previous section, the asset allocation result can also be derived directly in terms of put and call options. Although the resulting formula is more complex than (20), it is easy to understand because it has the form (3). Furthermore, in practice the payoff (20) can only be implemented by transacting in a discrete number of options, so a formulation which uses the options which are available in the market is important.

Since the stock itself can be regarded as a call option with zero strike, and because of put-call parity (8), the space of possible payoff functions $f(S)$ is spanned by $(S-K)^+$ for all $K \geqslant 0$ together with a constant payoff to represent cash. In what follows, sums will be written over $K \geqslant 0$, which should be understood as a sum over $K=0$ representing the stock itself together with the discrete option strikes which are being considered.

Let w_K be the weight for asset $(S-K)^+$ in a portfolio P that also includes an amount of cash w_c. Since the value of the options is determined by the risk-neutral probability distribution, the initial value P_0 of the portfolio is

$$P_0 = w_c + \sum_{K \geqslant 0} w_K \frac{E_{RN}\left((S-K)^+\right)}{(1+r_f\tau)} \quad . \tag{25}$$

Note that (4) implies that $E_{RN}\left((S-K)^+\right)/(1+r_f\tau)$ equals S_0 when $K=0$. Then since cash grows at the risk-free rate r_f, the expected value of P after time τ in the real-world is given by

$$E_{RW}(P) = w_c(1+r_f\tau) + \sum_{K \geqslant 0} w_K E_{RW}\left((S-K)^+\right) \quad , \tag{26}$$

with variance σ_P^2 given by

$$\sigma_P^2 = E_{RW}\left((P - E_{RW}(P))^2\right) = \sum_{K_1 \geqslant 0, K_2 \geqslant 0} w_{K_1} w_{K_2} \Sigma_{K_1 K_2}$$

where

$$\Sigma_{K_1K_2} = E_{RW}\left((S-K_1)^+(S-K_2)^+\right) - E_{RW}\left((S-K_1)^+\right)E_{RW}\left((S-K_2)^+\right) \quad . \tag{27}$$

Note that σ_P^2 does not depend on w_c because r_f is assumed here to be deterministic.

As in the previous section, the expected excess return is given by

$$\frac{E_{RW}(P) - E_{RN}(P)}{P_0} \tag{28}$$

so to work out the optimal weights, seek to maximise the utility function $U(w_c, w_K)$ defined by

$$U(w_c, w_K) = \frac{E_{RW}(P) - E_{RN}(P)}{P_0} - \frac{1}{2\lambda P_0^2}\sigma_P^2 \quad . \tag{29}$$

Differentiating $U(w_c, w_K)$ with respect to w_K shows that the maximum is given by

$$w_K = P_0 \lambda \sum_{K' \geqslant 0} (\Sigma_{KK'})^{-1} \mu_{K'} \quad , \tag{30}$$

where μ_K is defined by

$$\mu_K = E_{RW}\left((S-K)^+\right) - E_{RN}\left((S-K)^+\right) \quad . \tag{31}$$

and where w_c is whatever amount of cash needs to be borrowed or lent to set the initial portfolio value to P_0. Note that $K=0$ in (31) implies $\mu_{K=0} = \mu_S$ which is consistent with section 2.

Equation (31) shows how to generalise the definition of excess return to apply to options on a stock S. The answer is that the excess return μ_K is the difference between the real-world and risk-neutral expected values. Furthermore, the result (30) is completely analogous to the original Markowitz result (3), and

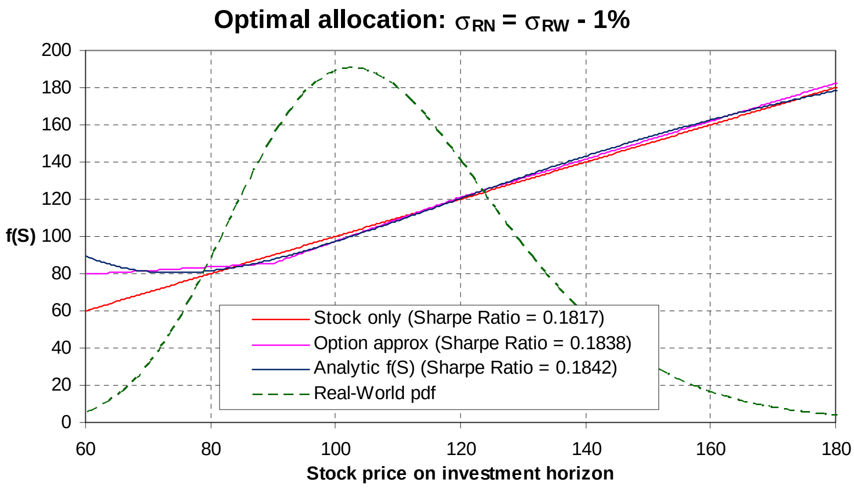

Figure 1: If the risk-neutral probability distribution has a volatility a bit lower than the real-world probability distribution, options are cheap relative to their expected payout and the optimal $f(S)$ contains bought option positions.

consistency is achieved between all the different assets $(S-K)^+$ because all the expectations in (27) and (31) are calculated using the same probability distribution for S.

In practice, as long as enough strikes K are chosen to span the payoff space, the payoff function implied by (30) is just an approximation to the continuous result which is given by (20). This will be illustrated below.

Note that the mean-variance formulation here uses notional sizes w_K for each option position. An alternative formulation in terms of the percentage of the initial portfolio value which is assigned to each option does not work because put-call parity is not preserved.

6 Using the asset allocation result in practice

To illustrate using the formulæ (20) and (30), a stock with the following properties will be used:

Price at start of investment period S_0	100.00
Budget P_0	100.00
Risk-free rate r_f	5.00%
Excess return rate μ_S	4.00%
Investment period τ (in years)	1.00
Forward F (risk-neutral mean)	105.00
Real-world mean $F + S_0\mu_S\tau$	109.00
Log-normal real-world volatility σ_{RW}	20.0%

. (32)

6.1 Log-normal distribution for S

Consider first the case when S is log-normally distributed. Choose the risk-tolerance parameter λ to be

$$\lambda = \frac{\left(1+\left(r_f+\mu_S\right)\tau\right)^2\left(\exp\left(\sigma_{RW}^2\tau\right)-1\right)}{\mu\tau} \tag{33}$$

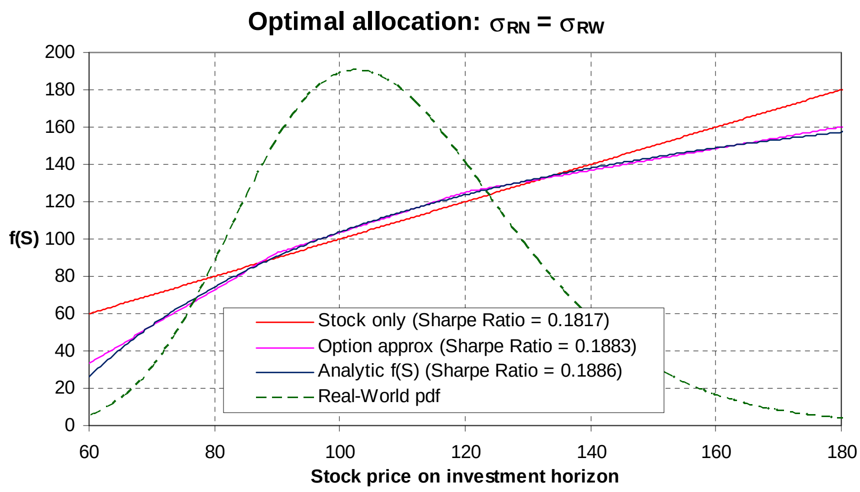

Figure 2: If the risk-neutral probability distribution has a volatility equal to the real-world probability distribution, then with log-normal distributions the optimal $f(S)$ is slightly short out-of-the-money options. Note that selling out-of-the-money options is not optimal with the actual left-skewed probability distributions seen in the market.

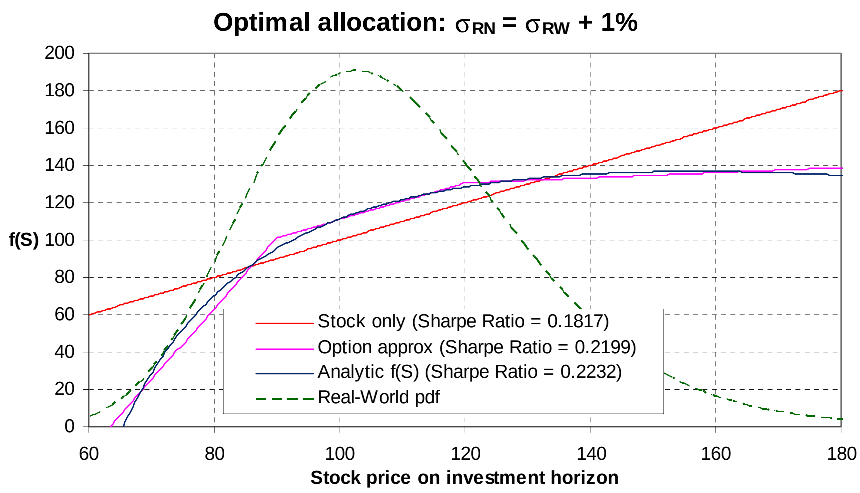

Figure 3: If the risk-neutral probability distribution has a volatility a bit higher than the real-world probability distribution, options are expensive relative to their expected payout and the optimal $f(S)$ contains even more sold option positions than in the case where.$\sigma_{RN} = \sigma_{RW}$ shown in figure 2.

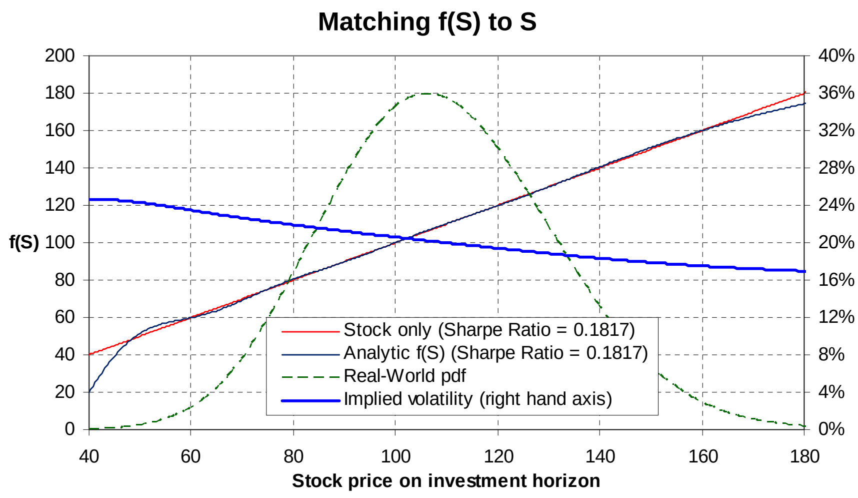

Figure 4: With a slight implied volatility skew, shown in the graph in blue, the optimal $f(S)$ is $f(S) = S$.

because with this λ, the optimal mean-variance allocation without options is a pure holding in the stock itself with no cash. Given the budget P_0 specified in (32), this means that without options the optimal holding would be a single unit of the stock S.

Using (60) from the appendix, figures 1, 2 and 3 show the optimal $f(S)$ for the cases where $\sigma_{RN} = \sigma_{RW} - 1\%$, $\sigma_{RN} = \sigma_{RW}$, and $\sigma_{RN} = \sigma_{RW} + 1\%$ respectively. In these graphs, the line labelled 'option approx' is the allocation calculated using (30) with the stock itself ($K = 0$) plus options at just two strikes ($K = 90$ and $K = 120$).

The figures 1, 2 and 3 illustrate three important points:

- The benchmark case where $\sigma_{RN} = \sigma_{RW}$ shown in figure 2 is slightly short out-of-the-money options. At the centre of the distribution the optimal strategy $f(S)$ does better than the 'Stock only' portfolio, but away from the centre the 'Stock only' portfolio does better. The implication of this is that if the real-world distribution is log-normal, an investor should sell out-of-the-money puts and out-of-the-money calls to optimise his risk versus reward. However it will be seen in section 6.3 below that in practice, the existence of a volatility skew in equity market volatilities changes this result so that buying out-of-the-money puts turns out to be the best strategy.
- The optimal allocation is sensitive to the relationship between the volatility used to price options σ_{RN} and the real-world volatility σ_{RW}. With $\sigma_{RN} = \sigma_{RW} - 1\%$ in figure 1, the optimal allocation has long option positions because the options are perceived to be cheap relative to their expected payout. Similarly with $\sigma_{RN} = \sigma_{RW} + 1\%$ in figure 3, the options are perceived to be expensive relative to their expected payout so the optimal allocation is even shorter options than the benchmark case in figure 2.
- The 'option approx' is replicating the optimal payoff $f(S)$ the best it can. The Sharpe Ratio for the 'option approx' is always higher than the 'Stock only' case, but not quite as high as what can be achieved by the optimal $f(S)$ in the continuous limit.

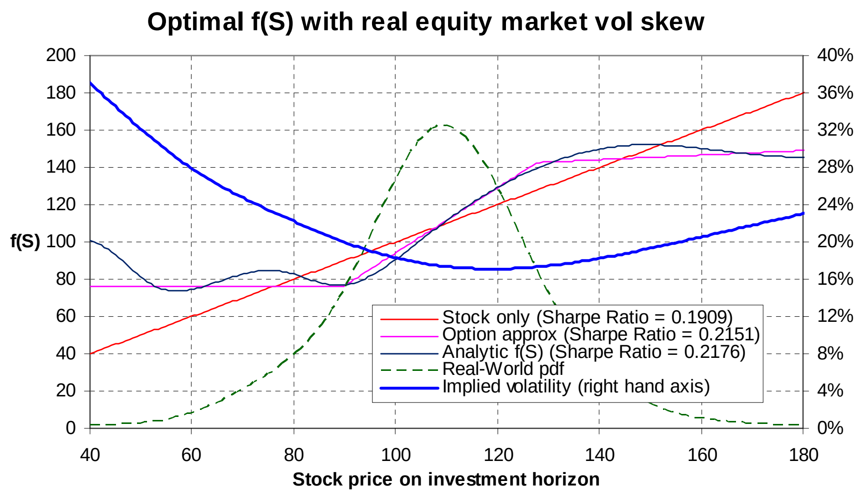

Figure 5: Although the optimal $f(S)$ (as shown by the wavy dark blue line) would be inpractical, the call spread shown by the pink line is practical and has almost the same Sharpe ratio as $f(S)$. The call spread is equivalent to a position in the underlying stock, together with a bought 15% delta put option and a sold 15% delta call option.

Using normal rather than log-normal distributions produces qualitatively the same results.

The fact that the above results are sensitive to the relationship between σ_{RN} and σ_{RW} emphasises the importance of the benchmark case in which ρ_{RN} and ρ_{RW} are the same apart from a difference in the mean. The result (20) spots when options are cheap or expensive relative to the uncertainty in the market, and buys or sells accordingly. The benchmark result where the only difference between ρ_{RN} and ρ_{RW} is the mean should consequently be regarded as the optimal strategy if the investor believes that options are fairly priced relative to the uncertainty in the market. Given that ρ_{RN} is known from (11), this means that the only additional information needed to construct the benchmark ρ_{RW} is μ, and determining μ has always been part of mean-variance asset allocation. As discussed in recent RBS asset allocation work[2], one source for μ is the equilibrium returns of the Capital Asset Pricing Model.

Beyond this benchmark allocation, if an investor then believes that market uncertainty is not correctly described by ρ_{RN}, then this implies a different ρ_{RW} and consequently a different allocation. Hence an investor who wants to take a view on market uncertainty can understand his allocation in terms of a base allocation where options correctly reflect market uncertainty, plus the effect of imposing a different uncertainty structure on the market compared to that implied by ρ_{RN}.

6.2 Probability distribution for a neutral option position

Given that a log-normal distribution implies that the optimal payoff $f(S)$ involves short positions in out-of-the-money put and call options, a natural question to ask is what probability distribution would give rise to a neutral option position in the benchmark case. In general there is no simple formula for the probability distribution ρ_{RN} which produces $f(S) = S$ from (20) where ρ_{RW} is ρ_{RN} with a shifted forward. However it is possible to solve this problem numerically, setting up ρ_{RN} as a function of several parameters, and then varying the parameters so as to minimise

$$\int (f(S) - S)^2 \rho_{RW}(S)\, dS \quad . \tag{34}$$

Figure 4 shows the result obtained using a probability distribution which was parameterised by a mixture of four log-normal distributions, where ρ_{RW} was constrained to have the same variance as a log-normal distribution with $\sigma_{RW} = 20\%$, and where the other parameters are specified by (32). The optimal $f(S)$ matches S very well at the centre of the distribution, and by adding more log-normal distributions a better fit would be possible at the edges of the distribution. The result is a probability distribution with a small option skew where lower strikes have slightly higher volatilities.

6.3 Real-world probability distribution

The option skew observed in the world equity markets is typically bigger than the skew shown in figure 4. Moving from the flat implied volatility curve of 20% which was used in figure 2 to the small volatility skew in figure 4 flattens out the short position in out-of-the-money put options, so the intuition is that moving further to a typical equity market volatility skew will give rise to an optimal $f(S)$ with a long position in out-of-the-money put options. This is indeed the case, as shown in figure 5.

The option volatility data which was used to construct the ρ_{RN} used in figure 5 came from the UK FTSE option market in July 2007. The data was fitted to a mixture of 3 log-normal distributions. The optimal $f(S)$, shown by the wavy blue line in the figure, can be well approximated by a simple call spread which is the pink line. Using put-call parity, the call spread is equivalent to a position in the stock, together with a long position in the 15% delta put option and a short position in the 15% delta call option. This has a Sharpe ratio of 0.215, whereas a pure position in the index has a lower Sharpe ratio of 0.191. Note that this allocation provides downside protection via the bought put option, as well as the almost optimal Sharpe ratio.

7 Beyond the mean-variance result

The example results presented in the previous section are encouraging. The main result (20) correctly goes longer options as options become cheap, and shorter options as they become more expensive. Furthermore, as the weight of the probability density function shifts, the optimal allocation shifts in sympathy. With the real-market data example shown in figure 5, there are more put options in the portfolio compared with figure 4 because there is more mass of the probability density function on the downside.

However it is natural to ask what the effect would be of adding an aversion to higher moments of $\rho_{RW}(S)$ into the utility functional $U[f]$. Because the result (20) is the solution to the linear equation (17), higher moments such as skewness and kurtosis can be incorporated into the framework analytically. To illustrate this, the next section 7.1 extends the result with a term which introduces an aversion to skewness. The result is a formula for $f(S)$ which is the solution of a quadratic equation. It will be shown, though, that the quadratic equation can often have complex roots. However, it will be seen that extending two of the examples from the previous section to incorporate a skewness aversion to the maximum possible extent while avoiding complex roots does not make a significant difference to $f(S)$.

It would also be possible to investigate kurtosis, and since that would involve solving a cubic equation, solutions will always be possible without complex roots. Instead of following that path though, section 7 focusses on whether a mean-variance approach is appropriate for portfolios which involve options. Looking at the probability distribution of the payoff $f(S)$ suggests that variance, and by extension higher moments, don't describe the distribution very well. This calls into question the entire approach taken by this article. Nonetheless, as discussed in the first paragraph of this section, the mean-variance result (20) seems to have all the right properties.

7.1 Incorporating an aversion to skewness

Adding a skewness term into the utility functional $U[f]$, with a parameter γ to describe aversion to skew, (16) becomes

$$U[f] = \int \frac{f(S)}{P_0}\left(\rho_{RW}(S) - \rho_{RN}(S)\right) dS - \frac{1}{2\lambda P_0^2}\left(\int f(S)^2 \rho_{RW}(S)\, dS - \left(E_{RW}(f(S))\right)^2\right) + \frac{\gamma}{3P_0^3}\int \left(f(S) - E_{RW}(f(S))\right)^3 \rho_{RW}(S)\, dS \quad . \tag{35}$$

The first order derivative condition is now

$$\rho_{RW}(S) - \rho_{RN}(S) - \frac{1}{\lambda P_0}\left(f(S)\rho_{RW}(S) - \rho_{RW}(S) E_{RW}(f(S))\right) + \frac{\gamma}{P_0^2}\left(\left(f(S) - E_{RW}(f(S))\right)^2 \rho_{RW}(S) - C_2 P_0^2\right) = 0 \quad . \tag{36}$$

where C_2 is a constant defined by

$$C_2 = \int \frac{\left(f(S) - E_{RW}(f(S))\right)^2}{P_0^2}\rho_{RW}(S)\, dS \quad . \tag{37}$$

Equation (36) is quadratic in $f(S)$. To see this more clearly, write

$$x = \frac{f(S) - E_{RW}(f(S))}{P_0} \tag{38}$$

so that (36) becomes

$$1 - \frac{\rho_{RN}(S)}{\rho_{RW}(S)} - \frac{x}{\lambda} + \gamma\left(x^2 - C_2\right) = 0 \quad . \tag{39}$$

The solution to this quadratic equation is

$$x = \frac{1}{2\gamma}\left(\frac{1}{\lambda} \pm \sqrt{\frac{1}{\lambda^2} - 4\gamma\left(1 - \frac{\rho_{RN}(S)}{\rho_{RW}(S)}\right) + 4\gamma^2 C_2}\right) \quad . \tag{40}$$

If a kurtosis term was introduced into $U[f]$, with a corresponding coefficient expressing aversion to kurtosis, then the resulting equation to solve would be cubic which can also be done analytically.

In theory, solutions for $f(S)$ are possible where one chooses '$+$' in (40) for some S and the '$-$' for other S. However, in general, such a choice would make $f(S)$ discontinuous, and it is also not clear whether both the '$+$' and '$-$' choices produce maxima of $U[f]$. If one imposes the condition that $f(S)$ must be continuous, then the same sign choice must be made for all S and since (38) implies that $E_{RW}(x) = 0$, the '$\pm$' in (40) must be '$-$' because choosing '$+$' all terms in (40) would be positive resulting in $E_{RW}(x) > 0$. This choice is also necessary to make the result reduce to (20) in the limit $\gamma \to 0$. Re-arranging (40) and writing in terms of $f(S)$ shows that

$$f(S) = E_{RW}(f(S)) + \frac{P_0}{2\lambda\gamma}\left(1 - \sqrt{1 + 4\lambda^2\gamma^2 C_2 - 4\lambda^2\gamma\left(1 - \frac{\rho_{RN}(S)}{\rho_{RW}(S)}\right)}\right) . \tag{41}$$

Taking the real-world expectation of (41) shows that C_2 must satisfy

$$\int \rho_{RW}(S)\sqrt{1 + 4\lambda^2\gamma^2 C_2 - 4\lambda^2\gamma\left(1 - \frac{\rho_{RN}(S)}{\rho_{RW}(S)}\right)}dS = 1 . \tag{42}$$

In practice, numerical methods must be used to solve (42) and hence determine C_2. However it is straightforward to show that if (42) is true, then (37) is also true. Then imposing the constraint that the portfolio is worth P_0 at the start of the investment period means that (41) can be written

$$\begin{aligned} f(S) = {} & P_0(1 + r_f\tau) \\ & + \frac{P_0}{2\lambda\gamma}\left(\int \rho_{RN}(S)\sqrt{1 + 4\lambda^2\gamma^2 C_2 - 4\lambda^2\gamma\left(1 - \frac{\rho_{RN}(S)}{\rho_{RW}(S)}\right)}dS\right. \\ & \left. - \sqrt{1 + 4\lambda^2\gamma^2 C_2 - 4\lambda^2\gamma\left(1 - \frac{\rho_{RN}(S)}{\rho_{RW}(S)}\right)}\right) . \end{aligned} \tag{43}$$

Equation (43) is the extension of (20) which incorporates an aversion to skewness. In the limit as $\gamma \to 0$, (43) can be written

$$\begin{aligned} f(S) = {} & E_{RW}(f(S)) + P_0\lambda\left(1 - \frac{\rho_{RN}(S)}{\rho_{RW}(S)}\right) \\ & + P_0\lambda\gamma\left(\lambda^2\left(1 - \frac{\rho_{RN}(S)}{\rho_{RW}(S)}\right)^2 - C_2\right) + O(\gamma^2) , \end{aligned} \tag{44}$$

which reduces to (20) when $\gamma = 0$. Hence for small γ,

$$C_2 = \lambda^2 \int \left(1 - \frac{\rho_{RN}(S)}{\rho_{RW}(S)}\right)^2 \rho_{RW}(S)\,dS + O(\gamma) . \tag{45}$$

If $\rho_{RN}(S) = \rho_{RW}(S)$ then (42) shows that $C_2 = 0$, and hence (43) proves that $f(S) = P_0(1 + r_f\tau)$ which is a pure cash position. Equation (43) also shows that the expected excess return is given by

$$\frac{E_{RW}(f(S))}{P_0} - (1 + r_f\tau) = \frac{\int \rho_{RN}(S)\sqrt{1 + 4\lambda^2\gamma^2 C_2 - 4\lambda^2\gamma\left(1 - \frac{\rho_{RN}(S)}{\rho_{RW}(S)}\right)}dS - 1}{2\lambda\gamma} . \tag{46}$$

However, one problem with (43) is that the argument of the square-root is not guaranteed to be positive. Even in the case where $\rho_{RN}(S)$ and $\rho_{RW}(S)$ are log-normal with $\sigma_{RN} = \sigma_{RW}$, for sufficiently large γ the result becomes complex for large S. The formulas that produced (62) show that in this case, $\rho_{RN}(S)/\rho_{RW}(S) \to 0$ as $S \to \infty$, so given a numerical solution for C_2 the argument of the square-root remains positive for all S if and only if

$$1 + 4\lambda^2\gamma^2 C_2 > 4\lambda^2\gamma . \tag{47}$$

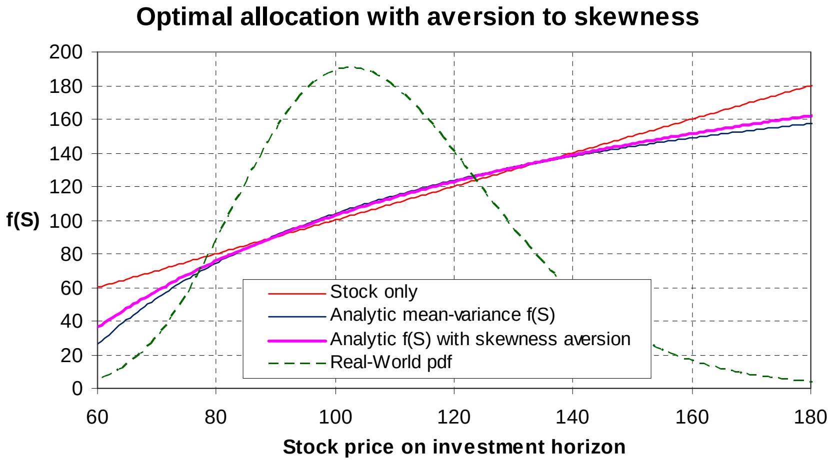

Figure 6: The effect of aversion to skewness on the case shown in figure 2, with the skewness aversion parameter $\gamma = 0.1716$.

Taking the example (32) with risk tolerance given by (33), (47) is satisfied when $\gamma < 0.1716$ but when $\gamma > 0.1717$ the solution becomes complex for sufficiently large S. Figure 6 shows the effect of introducing a skewness term with $\gamma = 0.1716$. Because there is a bit more aversion to risk, the optimal $f(S)$ is slightly less short out-of-the-money options when compared to the pure mean-variance result shown in figure 2. However, the qualitative result is that in this case, introducing an aversion to skewness does not have a big effect on the optimal $f(S)$. Similarly, in the real-world example shown in figure 5, introducing an aversion to skewness to the maximum extent possible while avoiding a complex $f(S)$ also does not make a significant difference to the optimal $f(S)$.

7.2 Probability distribution of the optimal payoff

When portfolios are comprised of stocks, variance seems like a reasonable measure of risk[1]. Although the true probability distribution of stock price returns is unknown, the distribution is surely unimodal, and the distributions of the returns on the optimal mean-variance portfolio will be unimodal too. However when considering options, although the distribution of the underlying stock is unimodal, the probability distribution of the optimal payoff need not be unimodal and so it is not clear that variance is still a reasonable measure of risk. Furthermore, if the distribution of the payoff is not unimodal, considering skewness and kurtosis in addition to variance does not solve the problem.

Given $\rho_{RW}(S)$, for any given payoff function $f(S)$ the real-world distribution $\rho_f(x)$ of the payoff on the investment horizon is given by

$$\rho_f(x) = \int \delta\left(f(S) - x\right) \rho_{RW}(S)\, dS \tag{48}$$

where $\delta(x)$ is the Dirac delta function. In the case where $f(S)$ is monotonic, then $f(S)$ has a unique inverse f^{-1}, in which case (48) can be written

$$\rho_f(x) = \frac{\rho_{RW}\left(f^{-1}(x)\right)}{f'\left(f^{-1}(x)\right)} \quad . \tag{49}$$

[1]The author is indebted to Riccardo Rebonato, Head of Market Risk & Head of Quantitative Research at RBS, for inspiring this section.

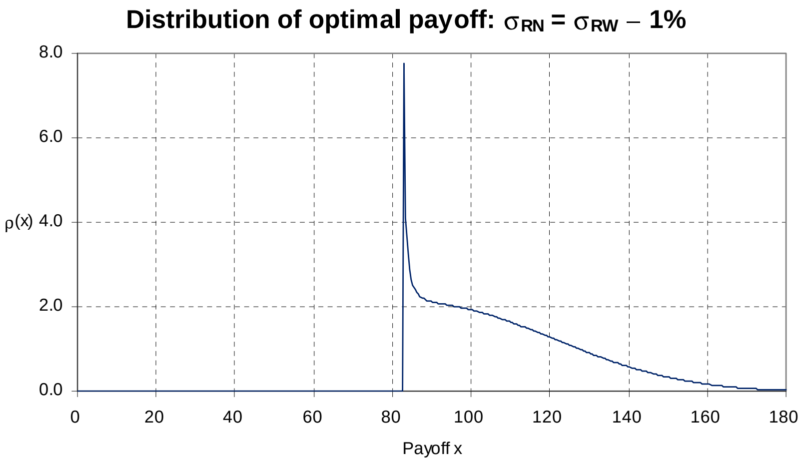

Figure 7: Distribution of optimal payoff $f(S)$ when S is log-normally distributed and when $\sigma_{RN} = \sigma_{RW} - 1\%$. In this case options are cheap, and $f(S)$ has a minimum with a value around 80. This is the $\rho_f(x)$ corresponding to figure 1.

In general, stationary points in $f(S)$ will give rise to peaks in $\rho_f(x)$. To understand why $\rho_f(x)$ is important, given (48) note that

$$E_{RW}(f(S)) = E(x) = \int x\rho_f(x)\,dx \quad , \tag{50}$$

$$\int (f(S) - E_{RW}(f(S)))^2 \rho_{RW}(S)\,dS = \int (x - E(x))^2 \rho_f(x)\,dx \quad . \tag{51}$$

Hence the utility functional $U[f]$ in (16) contains the mean and variance of the distribution $\rho_f(x)$, so it's the mean versus variance of $\rho_f(x)$ that the procedure of section 4 is optimising. All available distributions $\rho_f(x)$ are considered, and the resulting $f(S)$ from (20) implies the best distribution via (48).

As an example of (48), consider the probability distribution of the payoff of a call option. The payoff $f_K(S)$ on the investment horizon for a call option with strike K is given by

$$f_K(S) = (S - K)^+ - \chi_K \quad , \quad \text{where} \quad \chi_K = C(K)(1 + r_f\tau) \quad . \tag{52}$$

χ_K is the price of the call option at the start of the investment period rolled forward to the investment horizon at the risk-free interest rate r_f. Hence substituting (52) into (48) shows that

$$\rho_{f_K}(x) = \delta(x + \chi_K)\int_0^K \rho_{RW}(S)\,dS + \theta(\chi_K + x)\,\rho_{RW}(x + \chi_K + K) \quad , \tag{53}$$

where $\theta(x)$ is the Heaviside step function (7). Equation (53) shows that the distribution for a call option contains a delta function, corresponding to the range of S where the option is out-of-the-money at expiry. Hence if S_P is such that $\rho_{RW}(S_P)$ is the maximum value of $\rho_{RW}(S)$, then if $K < S_P$ the call option payoff distribution $\rho_{f_K}(x)$ is bimodal because it contains one peak at $x = S_P - K - \chi_K$ and another at the delta function when $x = -\chi_K$.

7.2.1 Distribution of payoff $\mathbf{f(S)}$ when $\mathbf{S}$ is log-normally distributed

Section 6.1 showed sample results for three cases where S is log-normally distributed. Figures 7, 8 and 9 show the $\rho_f(x)$ corresponding to 1, 2 and 3

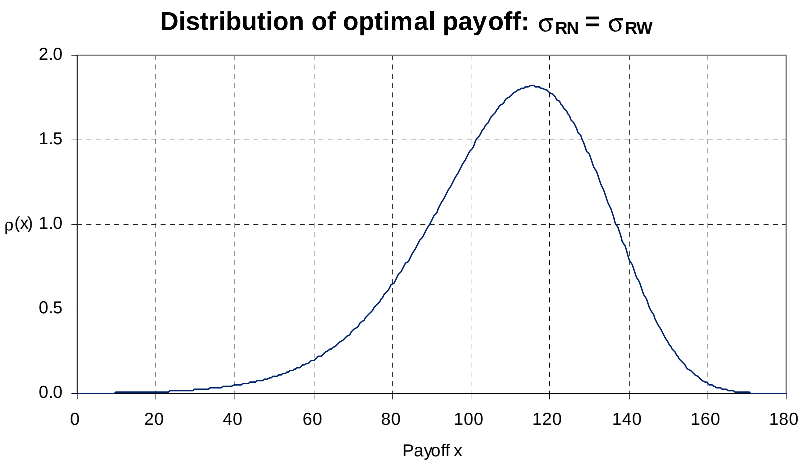

Figure 8: Distribution of optimal payoff $f(S)$ when S is log-normally distributed and when $\sigma_{RN} = \sigma_{RW}$. This is the $\rho_f(x)$ corresponding to figure 2.

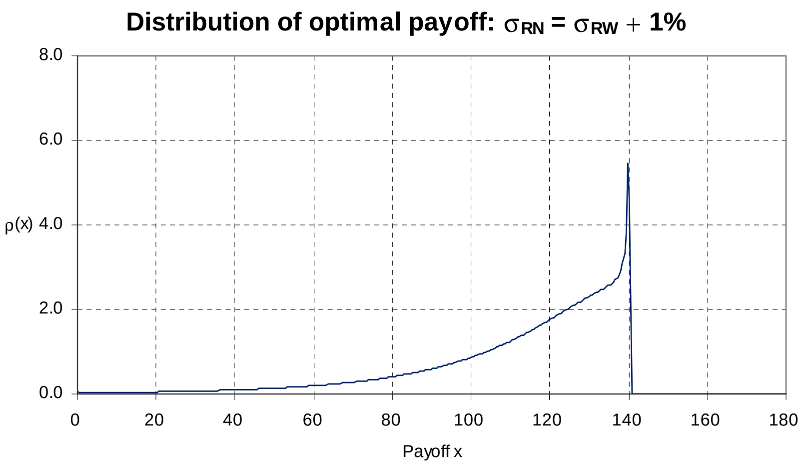

Figure 9: Distribution of optimal payoff $f(S)$ when S is log-normally distributed and when $\sigma_{RN} = \sigma_{RW} + 1\%$. In this case options are expensive, and $f(S)$ has a maximum with a value around 140. This is the $\rho_f(x)$ corresponding to figure 3.

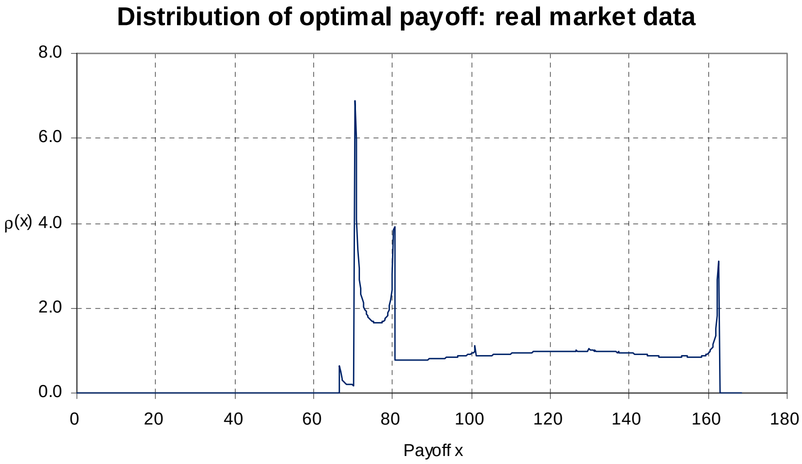

Figure 10: Distribution of optimal payoff $f(S)$ using real-market data. All the spikes in $\rho_f(x)$ correspond to maxima and minima of $f(S)$. This is the $\rho_f(x)$ corresponding to figure 5.

when $\sigma_{RN} = \sigma_{RW} - 1\%$, $\sigma_{RN} = \sigma_{RW}$ and $\sigma_{RN} = \sigma_{RW} + 1\%$ respectively. Although these distributions are all unimodal, only the distribution corresponding to $\sigma_{RN} = \sigma_{RW}$ looks as though it is reasonably well described by a mean and a variance. In the other two cases, the peaks of the distribution are at the edges, corresponding to the location of the maximum or minimum values of $f(S)$.

7.2.2 Distribution of payoff $\mathbf{f(S)}$ with real market data

With the real market data case shown in figure 5 , $f(S)$ has several maxima and minima, and figure 10 shows $\rho_f(x)$ for this case. The peaks of $\rho_f(x)$ correspond to the maxima and minima of $f(S)$. As for the log-normal cases above where $\sigma_{RN} = \sigma_{RW} - 1\%$ and $\sigma_{RN} = \sigma_{RW} + 1\%$, mean and variance provide poor descriptions of this distribution.

8 Conclusion

Traditional Markowitz mean-variance has been extended to incorporate options. The main results are the optimal payoff function $f(S)$ given by (20) in section 4, and the option oriented approximation to (20) given by (30) in section 5.

Section 6 analysed the result in detail. The benchmark case that an investor should consider is where the real-world distribution is derived from the risk-neutral distribution by adjusting the mean of the distribution for the expected excess return.

In the theoretical cases where returns are either normally or log-normally distributed, the optimal payoff is associated with short positions in out-of-the-money options. Beyond that, the probability distribution where the optimal payoff is a pure position in the underlying stock corresponds to a situation where there is a volatility skew with lower option strikes having a slightly higher volatility. However in the real-world, the volatility skew is typically bigger than that, and the result is an allocation which is long out-of-the-money put options. Using sample data from the UK FTSE index from July 2007, it was shown that the optimal position is close to a call spread on the index. This provides downside protection on the index, as well as the almost optimal Sharpe ratio.

Section 7 considers issues beyond the basic mean-variance result. It was shown that skewness can be incorporated easily, although in the two examples considered, it didn't have a big effect on the result. It would also be relatively easy to incorporate kurtosis. However section 7.2 considered the probability distribution of the payoff, and showed that this approach which considers moments of the probability distribution $\rho_{RW}(x)$ may not be appropriate in the presence of options.

Nonetheless, the mean-variance result (20) seems to have all the right properties. The optimal payoff function $f(S)$ correctly goes longer options as options become cheap, and shorter options as they become more expensive. Furthermore, examples have been presented to show that as the weight of the probability density function shifts, the optimal allocation shifts in sympathy. It should also be relatively easy to extend these results to analyse the case where there are multiple stocks together with options on all the stocks.

Appendix: Optimal $f(S)$ for normal and log-normal distributions

The formulæ in sections 4 and 5 are general in the sense that the probability distributions for S were not specified. Although in reality asset returns are neither normally nor log-normally distributed, nonetheless it is instructive to analyse these cases to develop intuition relating to how the result (20) works. This appendix first gives the formula for the optimal $f(S)$ in the cases where the distribution for S is either normal or log-normal, followed by the corresponding formula required to implement the discrete approximation (30).

If the returns of the stock S are either normally or log-normally distributed, it is straightforward to derive the optimal $f(S)$. With normal distributions, if

$$\text{Risk-Neutral } \rho_{RN}(S): \quad S \sim N\left(F, \sigma_{RN}^2 \tau\right) \tag{54}$$

$$\text{Real-World } \rho_{RW}(S): \quad S \sim N\left(F + S_0 \mu_S \tau, \sigma_{RW}^2 \tau\right) \tag{55}$$

then the optimal $f(S)$ as specified by (20) is given by

$$f(S) = P_0 \left(1 + r_f \tau\right) + P_0 \lambda \frac{\sigma_{RW}}{\sigma_{RN}} \left\{ \frac{\sigma_{RW}}{\sqrt{2\sigma_{RW}^2 - \sigma_{RN}^2}} \exp\left(\frac{S_0^2 \mu_S^2 \tau}{2\sigma_{RW}^2 - \sigma_{RN}^2} \right) - \exp\left(\frac{\left(S - F - S_0 \mu_S \tau\right)^2}{2\sigma_{RW}^2 \tau} - \frac{\left(S - F\right)^2}{2\sigma_{RN}^2 \tau} \right) \right\} . \tag{56}$$

In the special case where $\sigma_{RN} = \sigma_{RW} = \sigma$, so that the only difference between ρ_{RN} and ρ_{RW} is the mean, (56) reduces to

$$f(S) = P_0 \left(1 + r_f \tau\right) + P_0 \lambda \left\{ \exp\left(\frac{S_0^2 \mu^2 \tau}{\sigma^2} \right) - \exp\left(\frac{S_0^2 \mu_S^2 \tau + 2\left(F - S\right) S_0 \mu_S}{2\sigma^2} \right) \right\} , \tag{57}$$

which has the form of $\alpha - \beta e^{-\gamma S}$ for positive constants α, β, γ which can be determined from (57). The formulæ needed to implement (30) when S is normally distributed are given below.

With log-normal distributions, if

$$\text{Risk-Neutral } \rho_{RN}(S): \quad \ln(S) \sim N\left(\ln(F) - \frac{\sigma_{RN}^2 \tau}{2}, \sigma_{RN}^2 \tau\right) \tag{58}$$

$$\text{Real-World } \rho_{RW}(S): \quad \ln(S) \sim N\left(\ln(F + S_0 \mu_S \tau) - \frac{\sigma_{RW}^2 \tau}{2}, \sigma_{RW}^2 \tau\right) \tag{59}$$

then the optimal $f(S)$ as specified by (20) is given by

$$f(S) = P_0(1 + r_f \tau) + P_0 \lambda \frac{\sigma_{RW}}{\sigma_{RN}} \left\{ \frac{\sigma_{RW}}{\sqrt{2\sigma_{RW}^2 - \sigma_{RN}^2}} \exp\left(\frac{\left(\mu_S' + \frac{1}{2}\left(\sigma_{RN}^2 - \sigma_{RW}^2\right)\right)^2 \tau}{\left(2\sigma_{RW}^2 - \sigma_{RN}^2\right)} \right) \right.$$
$$\left. - \exp\left(\frac{\left(\ln\left(\frac{S}{F + S_0 \mu_S \tau}\right) + \frac{\sigma_{RW}^2 \tau}{2}\right)^2}{2\sigma_{RW}^2 \tau} - \frac{\left(\ln\left(\frac{S}{F}\right) + \frac{\sigma_{RN}^2 \tau}{2}\right)^2}{2\sigma_{RN}^2 \tau} \right) \right\} , \tag{60}$$

where μ_S' is defined by

$$\mu_S' = \frac{1}{\tau} \ln\left(\frac{1 + (r_f + \mu_S)\tau}{1 + r_f \tau} \right) . \tag{61}$$

In the special case where $\sigma_{RN} = \sigma_{RW} = \sigma$, (60) reduces to

$$f(S) = P_0(1 + r_f \tau) + P_0 \lambda \left\{ \exp\left(\frac{\mu_S'^2 \tau}{\sigma^2} \right) \right.$$
$$\left. - e^{-\frac{1}{2}\mu' \tau} \left(\frac{S}{S_0 \sqrt{(1 + r_f \tau)(1 + (r_f + \mu)\tau)}} \right)^{\left(-\frac{\mu_S'}{\sigma^2}\right)} \right\} , \tag{62}$$

which has the form $\alpha - \beta S^{-\gamma}$ for positive constants α, β, γ which can be determined from (62). The formulæ needed to implement (30) when S is log-normally distributed are given below.

Discrete formulation: normal distribution for S.

If the forward distribution of S is normally distributed with mean F, then after a time period τ

$$S = F + x\,\sigma\sqrt{\tau} \quad \text{with probability} \quad \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} . \tag{63}$$

Then the forward option value $E\left((S-K)^+\right)$ is

$$E\left((S-K)^+\right) = (F-K)\,N(h) + \frac{\sigma\sqrt{\tau}}{\sqrt{2\pi}} e^{-\frac{1}{2}h^2} , \tag{64}$$

where

$$h = \frac{F-K}{\sigma\sqrt{\tau}} , \tag{65}$$

and the second moment matrix $E\left((S-K_1)^+ (S-K_2)^+\right)$ is given by

$$E\left((S-K_1)^+ (S-K_2)^+\right) = \left((F-K_1)(F-K_2) + \sigma^2 \tau\right) N(h_{\max})$$
$$+ \frac{\sigma\sqrt{\tau}\,(F - \min(K_1, K_2))}{\sqrt{2\pi}} e^{-\frac{1}{2}h_{\max}^2} , \tag{66}$$

where

$$h_{\max} = \frac{F - \max(K_1, K_2)}{\sigma\sqrt{\tau}} . \tag{67}$$

Discrete-formulation: log-normal distribution for S

If the forward distribution of S is log-normally distributed with mean F, then after a time period τ

$$S = F \exp\left(x\,\sigma\sqrt{\tau} - \frac{1}{2}\sigma^2\tau\right) \quad \text{with probability} \quad \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2} \quad . \tag{68}$$

Then the forward option value $E\left((S-K)^+\right)$ is

$$E\left((S-K)^+\right) = FN\left(h + \sigma\sqrt{\tau}\right) - KN\left(h\right) \quad , \tag{69}$$

where

$$h = \frac{\ln\left(\frac{F}{K}\right)}{\sigma\sqrt{\tau}} - \frac{1}{2}\sigma\sqrt{\tau} \quad , \tag{70}$$

and the second moment matrix $E\left((S-K_1)^+(S-K_2)^+\right)$ is given by

$$\begin{aligned} E\left((S-K_1)^+(S-K_2)^+\right) &= F^2e^{\sigma^2\tau}N\left(h_{\max} + 2\sigma\sqrt{\tau}\right) + K_1K_2N\left(h_{\max}\right) \\ &\quad - F\left(K_1+K_2\right)N\left(h_{\max} + \sigma\sqrt{\tau}\right) \quad , \end{aligned} \tag{71}$$

where

$$h_{\max} = \frac{\ln\left(\frac{F}{\max(K_1,K_2)}\right)}{\sigma\sqrt{\tau}} - \frac{1}{2}\sigma\sqrt{\tau} \quad . \tag{72}$$

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[2] Paul Doust, "A geometric approach to stable asset allocation", RBS research article, 24th May 2007.